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Multiplication and combinatorics in the Steenrod algebra

Judith H. Silverman

Department of Mathematics, Indiana University, Purdue University at Columbus, 4601 Central Ave., Columbus, IN 47203 - 1769, USA

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Abstract

This paper is concerned with multiplication in the mod-2 Steenrod algebra, as expressed in terms of both the Milnor basis and the basis of admissible elements. Part I describes techniques for graphically representing Milnor basis elements and for interpreting combinatorially the matrices that arise in their multiplication table, and Part II provides a method for the simultaneous computation of families of Adem relations. Part III combines the techniques of Parts I and II to identify constraints on the Milnor basis elements which occur in the image under the canonical anti-automorphism of certain products.

PART I. MULTIPLICATION OF MILNOR BASIS ELEMENTS

1. Preliminaries

The Milnor basis of the mod-2 Steenrod algebra $\mathcal{A}(2)$ of cohomology operations is indexed by sequences $T = (t_1, t_2, \dots)$ of non-negative integers almost all of which are 0 [4]. If T is such a sequence for which $t_l = 0$ for $l > n$, we denote the corresponding basis element by $Sq(T) = Sq(t_1, \dots, t_n)$; its dimension is $|Sq(T)| = \sum_i (2^i - 1)t_i$. For $T = (t)$, the corresponding basis element $Sq(t)$ is the Steenrod square commonly denoted Sq^t. The product $Sq(a_1)\cdots Sq(a_n)$ is called *admissible* if $a_r \geq 2a_{r+1}$ for $r < n$ and $a_n > 0$ if $n > 1$. Admissible elements themselves form an (additive) basis of $\mathcal{A}(2)$ [7]. For both bases it is known how to express the product of two generators as a sum of other generators; these product formulas are discussed in Parts I and II respectively.

Both of these multiplication formulas involve binomial or multinomial coefficients taken mod 2. A well-known theorem due to Lucas gives a criterion for the binomial coefficient $\binom{n}{p}$ to be odd: Let $n_\tau \cdots n_1 n_0$ and $p_\sigma \cdots p_1 p_0$ be the binary representations

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of n and p respectively. We write $n \succcurlyeq_i p$ to mean $n_i \geq p_i$ and we say n *dominates* p $(n \succcurlyeq p)$ if $n_i \succcurlyeq_i p_i$ for all i (so that in particular, $n \geq p$). It is proven in [3] that the coefficient $\binom{n}{p}$ is odd $\iff n \succcurlyeq p$ $[\iff n \succcurlyeq n-p]$. In other words, each power of 2 appearing in the binary representation of n appears in exactly one of the binary representations of p and $n-p$. More generally, if $m \geq 3$ and $\sum_{i=1}^{m} p_i = n$, the multinomial coefficient giving the number of ways to divide a set of n elements into m subsets of orders $p_1, \ldots, p_m$ is written

$$\binom{n}{p_1 \mid p_2 \mid \ldots \mid p_m} = \binom{n}{p_1}\binom{n-p_1}{p_2}\binom{n-(p_1+p_2)}{p_3}\cdots \times \binom{n-(p_1+p_2+\cdots+p_{m-1})}{p_m}.$$

Lucas's theorem and an inductive argument imply that $\binom{n}{p_1 \mid p_2 \mid \ldots \mid p_m} \equiv 1 \pmod 2$ $\iff$ each power of 2 in the binary representation of n occurs in exactly one of the binary representations of $p_1,\ p_2,\ \ldots, p_m$.

The Steenrod algebra acts on $\mathbf{F}_2[x_1,\ldots,x_s]$, the polynomial algebra on s generators x_i of dimension 1, which is the mod-2 cohomology ring of $\overbrace{\mathbf{R}P^\infty \times \cdots \times \mathbf{R}P^\infty}^{s}$. The *excess* of an element $\theta \in \mathcal{A}(2)$ is given by $ex(\theta) = \min\{\hat{s} \ :\ \theta(x_1x_2\cdots x_{\hat{s}}) \neq 0 \in \mathbf{F}_2[x_1,\ldots,x_{\hat{s}}]\ \}$. In [2] , Kraines proves that $ex[Sq(T)] = \sum_{l=1}^{n} t_l$ and that the excess of a sum of Milnor basis elements is the minimum of the excesses of the summands.

The Steenrod algebra is a connected Hopf algebra, and as such has a unique antiautomorphism χ such that $\chi Sq(1) = Sq(1)$ and $\sum_{i=1}^{n} Sq(i)\chi Sq(n-i) = 0$. In [4], Milnor proves that $\chi Sq(w)$ is the sum of all Milnor basis elements of degree w. Kraines's theorem of the previous paragraph implies that $ex[\chi Sq(w)] = \mu(w)$, where $\mu(w)$ is the number of summands in the most efficient way of writing w as a sum of numbers of the form $2^k - 1$ [4]. That is, $\mu(w) = \min\{m \ :\ w = \sum_{i=1}^{m}(2^{k_i}-1)$ for some integers $k_i\}$.

In what follows, we will be writing numbers in *vertical binary*; i.e., in binary notation with powers of 2 increasing from top to bottom rather than from right to left.

2. Combs

2.1. Notation

Recall that the dimension of the Milnor element $Sq(T) = Sq(t_1,\ldots,t_p)$ is given by $|Sq(T)| = \sum_1^p (2^l - 1)t_l$. We interpret this dimension as the value of the sequence $T = (t_1,\ldots,t_p)$ in a system in which the lth term counts for $2^l - 1$ times its face value. Since $2^l - 1$ is written in vertical binary as a column of l 1's, we may represent the sequence with the picture

$$\begin{array}{lcl}
\overbrace{1\ 1\ 1\ \ldots\ 1\ 1}^{t_1}\ \overbrace{1\ 1\ 1\ \ldots\ 1\ 1}^{t_2} & & \overbrace{1\ 1\ 1\ \ldots\ 1\ 1}^{t_p} \\
\ 1\ 1\ 1\ \ldots\ 1\ 1 & & 1\ 1\ 1\ \ldots\ 1\ 1 \\
 & \cdots\cdots & \vdots\ \vdots\ \vdots\ \ddots\ \vdots\ \vdots \\
 & & 1\ 1\ 1\ \ldots\ 1\ 1
\end{array} \tag{1}$$

This picture, or any obtained from it by a permutation of columns, will be called the *comb* $C(T)$; a column of l 1's is a *tooth of length* l, and its value viewed as a vertical binary number, $2^l - 1$, is its *weight*. The *weight of a comb* is the combined weight of its teeth, which equals the dimension of the corresponding Milnor element. In view of Kraines's result, cited above, the number of teeth of a comb is the excess of the corresponding Milnor element; we define the *excess* of $C(T)$ to be $ex(Sq(T)) = \sum t_i$. The rows of a comb are numbered from the top, starting with the 0th.

Combs admit structures, difficult to describe in the world of Milnor elements, which simplify the task of determining whether a given Milnor element appears as a summand in the product of two others. These structures are described in the following sections.

2.2. Bundles

In order to discuss the multiplication of Milnor basis elements in terms of combs, it is convenient to be able to manipulate the teeth of a comb in groups as well as singly. A *bundle of size* 2^σ is a collection of 2^σ teeth of the same length. Such a bundle is represented in column form as a sort of generalized tooth having the same number of 1's as the teeth comprising it but preceded by a number of 0's corresponding to the power of 2 involved. Four teeth of length 3, for instance, have a combined weight of $4(2^3 - 1)$, which in binary notation is written

$$\begin{array}{c} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{array}.$$

In general, a column consisting of σ 0's above l 1's can be unambiguously identified as a bundle of 2^σ teeth of length l.

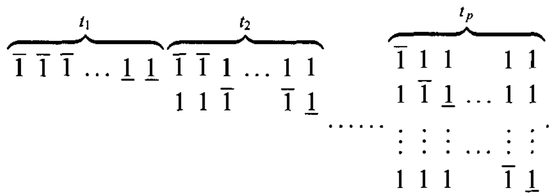
Let $T = (t_1, \ldots, t_p)$ as above, and let $C(T)$ be its comb. Any way of writing the t_l, $1 \le l \le p$, as sums of powers of 2 gives rise to a bundle structure on $C(T)$: if $t_l = 2^{\alpha_{1,l}} + 2^{\alpha_{2,l}} + \cdots + 2^{\alpha_{s,l}}$, then the teeth of length l are arranged in bundles of size $2^{\alpha_{1,l}}, \ldots, 2^{\alpha_{s,l}}$. The order of the bundles is not important. Of particular interest is the *canonically bundled comb* $C_b(T)$, in which the teeth of length l are bundled according to the binary representation of t_l. For example, the canonical comb $C_b(9,7,6,1)$ is

$$\begin{array}{llll} 0\ 1 & 0\ 0\ 1 & 0\ 0\ 1 \\ 0 & 0\ 1\ 1 & 0\ 1\ 1 \\ 0 & 1\ 1 & 1\ 1\ 1. \\ 1 & 1 & 1\ 1\ 1 \\ & & 1 \end{array}$$

A bundled comb encodes the same information about $Sq(T)$ as does the underlying comb $C(T)$. In particular, as a bundle of size 2^σ has its topmost 1 in the σth row, one obtains the excess of $Sq(T)$ by adding up the place values of the topmost 1's of the generalized teeth of any bundled comb for T.

2.3. Partitions

One more piece of structure is necessary to discuss Milnor multiplication in terms of combs. A *partitioned* comb is one in which each tooth is split in two horizontally. More precisely, it is a comb along with a choice, for each tooth τ of length l, of a *partition number* $0 \le \pi(\tau) \le l$ indicating that the tooth is to be split above the $\pi(\tau)$th row. Graphically we represent the partition number of each tooth as a horizontal "partition line" across the tooth:



The teeth of partitioned combs can be combined into bundles as before; the generalized teeth of a bundled partitioned comb represent sets of 2^σ teeth with the same length and partition number. Conversely, a bundled comb can be given a partition: in this case, each generalized tooth τ of size 2^σ and length l is assigned a partition number $0 \le \pi(\tau) \le l$ indicating that the generalized tooth is to be split above the $(\sigma + \pi(\tau))$th row.

A partition of the (ordinary) comb C is *compatible* with a bundle structure on C if one can indicate the partition on a picture of the bundled comb. Technically speaking, the condition is as follows: given integers $0 \le \pi \le l$, let N^l_π be the number of teeth of length l required by the partition structure to have partition number π. The partition is compatible with the bundle structure if the generalized teeth of each length l in the bundled comb can be arranged in l groups in such a way that the number of ordinary teeth represented in the πth group is N^l_π. Each generalized tooth in the πth group is then assigned the partition number π.

Example. If $T = (4)$, the partition of $C(T)$ given by

$$\overline{1}\;\overline{1}\;\underline{1}\;\underline{1}$$

is compatible with the first of the bundle structures pictured below, with the assignment of partition numbers to teeth as indicated, but is not compatible with the second of the bundle structures:

$$\begin{array}{cc} 0 & 0 \\ \overline{1} & \underline{1} \end{array} \qquad \begin{array}{c} 0 \\ 0 \\ 1 \end{array}.$$

Finally, we define the *first part* (resp. *second part*) of a partitioned bundled comb K to be the bundled comb obtained by replacing all the 1's of K below (resp. above) the partition lines with blanks (resp. 0's).

In our discussion of the Milnor product formula below, we shall see how the notions of bundled and partitioned combs simplify the work involved in determining if a given Milnor basis element occurs in the product of two others. From now on, all combs (resp. teeth) will be understood to be bundled (resp. generalized).

3. Multiplication

3.1. The Milnor product formula

Let $R = (r_1, \ldots, r_m)$, $S = (s_1, \ldots, s_n)$, and $T = (t_1, \ldots, t_p)$ be sequences of non-negative integers with $|Sq(R)| + |Sq(S)| = |Sq(T)|$. Our goal is to determine if $Sq(T)$ is a summand in the product $Sq(R)\cdot Sq(S)$, i.e., if $Sq(T)$ appears with coefficient 1 in the unique representation of θ as a sum of Milnor basis elements. In [4], Milnor describes the product in terms of certain matrices: Let $\mathcal{X}_{R,S}$ be the set of infinite matrices

$$X = \begin{pmatrix} * & x_{01} & x_{02} & \cdots \\ x_{10} & x_{11} & x_{12} & \cdots \\ x_{20} & x_{21} & x_{22} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

of non-negative integers with x_{00} omitted such that

$$\sum_{j=0}^{\infty} 2^j x_{ij} = r_i \quad \text{for all } i \geq 1, \tag{2}$$

$$\sum_{i=0}^{\infty} x_{ij} = s_j \quad \text{for all } j \geq 1. \tag{3}$$

For each matrix $X \in \mathcal{X}_{R,S}$, define the sequence $T(X) = (t_1, t_2, \ldots)$ by $t_l = \sum_{i=0}^{l} x_{i,l-i}$

and let $b_l(X)$ be the multinomial coefficient

$$\binom{n}{x_{0,l} \mid x_{1,l-1} \mid \ldots \mid x_{l,0}}.$$

Then the product $Sq(R)\cdot Sq(S)$ is given by

$$Sq(R)\cdot Sq(S) = \sum_{X \in \mathcal{X}_{R,S}} [b_1(X)b_2(X)\cdots]\ Sq[T(X)].$$

Thus $Sq(T)$ is a summand of $Sq(R)\cdot Sq(S) \iff a(T)$ is odd, where $a(T)$ is the number of matrices $X \in \mathcal{X}_{R,S}$ with $T(X) = T$ and $b_l(X) \equiv 1 \pmod 2$ for all l. Rather than trying to construct such matrices one by one, it is often advantageous to translate the question into the language of combs and invoke the bundle and partition structures described above.

3.2. Interpretation of Milnor's formula

Fix R and S, and suppose that $X \in \mathcal{X}_{R,S}$ with $T(X) = T$. The matrix X is associated not only to the comb $C(T)$ but also to the partitioned comb $PC_X(T)$ which for all i, j has $x_{i,j}$ teeth of length $i + j$ and partition number j.

Example. Let $R = (14, 15)$ and $S = (8,4)$. Below are pictured a matrix $X \in \mathcal{X}_{R,S}$ with $T(X) = (3,6,4,2)$ and the associated partitioned comb $PC_X(T)$.

$$\begin{pmatrix} * & 1 & 1 & 0 & \cdots \\ 2 & 4 & 1 & 0 & \cdots \\ 1 & 3 & 2 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \qquad \begin{array}{ccc c cccccc c cccc c cc} \underline{1} & \bar{1} & \bar{1} & & 1 & 1 & 1 & 1 & 1 & \bar{1} & & 1 & 1 & 1 & 1 & & 1 & 1 \\ & & & & \underline{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & 1 & & 1 & \bar{1} & \bar{1} & \bar{1} & & 1 & 1 \\ & & & & & & & & & & & \bar{1} & 1 & 1 & 1 & & \bar{1} & \bar{1} \\ & & & & & & & & & & & & & & & & 1 & 1 \end{array}.$$

Observe that $b_2(X) \equiv b_3(X) \equiv 0 \pmod 2$, and that the partition indicated is not compatible with the canonical bundle structure $C_b(T)$.

We view $PC_X(T)$ as the result of suspending bundles of teeth of $C(R)$ below teeth of $C(S)$ of appropriate length. That is, a tooth of length $i + j$ and partition number j is obtained from a tooth of $C(S)$ of length j by appending a bundle of 2^j teeth of length i of $C(R)$. Eqs. (2) and (3) guarantee that the first part of the partitioned comb $PC_X(T)$ is exactly the comb $C(S)$ and the second part of $PC_X(T)$ is a (bundled) comb for R. Conversely, any partition of $C(T)$ whose first and second parts are combs for S and R respectively is readily seen to be $PC_Y(T)$ for some $Y \in \mathcal{X}_{R,S}$ with $T(Y) = T$.

Recall now from Section 3.1 that $Sq(T)$ is a summand of $Sq(R)\cdot Sq(S) \iff$ there is an odd number $a(T)$ of matrices with $T(X) = T$ for which the multinomial coefficients $b_l(X) \equiv 1 \pmod 2$ for all l. In the context of partitioned matrices, these multinomial coefficients arise naturally. As discussed in Section 1, $b_l(X)$ is odd when each power of 2 in the binary representation of the sum $t_l = \sum_{i=0}^{l} x_{i,l-i}$ occurs in the binary representation of exactly one of the $x_{i,l-i}$. But these powers of 2 are exactly the sizes of the generalized teeth of length l in the canonical bundle $C_{\mathrm{b}}(T)$. Therefore the above condition may be rephrased as the requirement that the generalized teeth of length l of $C_{\mathrm{b}}(T)$ can be divided into l groups in such a way that the sizes of the teeth in the ith group add up to $x_{i,l-i}$. In the notation of Section 2.3, this is the case for all $l \iff$ the partition structure of $PC_X(T)$ is compatible with the canonical bundle structure on $C(T)$. Accordingly the number $a(T)$ is exactly the number of partitions of the canonical comb $C_{\mathrm{b}}(T)$ whose first parts are combs for S and second parts combs for R. This proves the following theorem:

Theorem 3.1. *With notation as above, the Milnor element $Sq(T)$ is a summand of the product $Sq(R)\cdot Sq(S) \iff$ there is an odd number of partitions of the canonical comb $C_{\mathrm{b}}(T)$ whose first parts are combs for S and whose second parts are (bundled) combs for R.*

Such partitions can be easily read from the picture of $C_{\mathrm{b}}(T)$ if the numbers involved are small or if T is of a special form.

Example. We illustrate the situation when $R = (8,0,0,1,1)$, $S = (0,2,0,2)$, and $T = (0,0,2,3,1)$. In this case there is only one partition of $C_{\mathrm{b}}(T)$ with appropriate first and second parts:

$$\begin{array}{cccc} 0 & \bar{1} & 0 & \bar{1} \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ \bar{1} & 1 & 1 & 1 \\ & & \underline{1} & 1 \end{array},$$

so that $Sq(T)$ is indeed a summand of $Sq(R)\cdot Sq(S)$. On the other hand, for $R' = (8,8,2,2)$, $S' = (1,2,1)$, and T as above, there are two partitions of $C_{\mathrm{b}}(T)$ with appropriate first and second parts:

$$\begin{array}{cccc} 0 & 1 & 0 & 1 \\ 1 & \bar{1} & \bar{1} & 1 \\ 1 & 1 & 1 & 1 \\ \bar{1} & 1 & 1 & \bar{1} \\ & & 1 & 1 \end{array} \quad \text{and} \quad \begin{array}{cccc} 0 & 1 & 0 & 1 \\ \bar{1} & 1 & 1 & \bar{1} \\ 1 & 1 & 1 & 1 \\ 1 & \bar{1} & \bar{1} & 1 \\ & & 1 & 1 \end{array},$$

so that $Sq(T)$ is not a summand of $Sq(R')\cdot Sq(S')$.

3.3. Multiplication and the canonical antiautomorphism

Let χ denote the canonical automorphism of $\mathcal{A}(2)$, and let r and s be integers. The comb method lends itself particularly well to identifying terms of the product $\chi[Sq(r)]\cdot\chi[Sq(s)]$. Recall from Section 1 that for any integer w, $\chi[Sq(w)]$ is the sum of all Milnor basis elements of dimension w. Thus an element $Sq(T)$ of dimension

$r+s$ is a summand of $\chi[Sq(r)] \cdot \chi[Sq(s)] \iff Sq(T)$ is a summand of $Sq(R')\cdot Sq(S')$ for an odd number of pairs $(Sq(R'), Sq(S'))$ of Milnor elements of dimensions r and s respectively. In terms of combs, the condition is as follows:

Theorem 3.2. *With notation as above, the Milnor element $Sq(T)$ is a summand of the product $\chi[Sq(r)] \cdot \chi[Sq(s)] \iff$ there is an odd number of partitions of the canonical comb $C_b(T)$ whose first parts have weight s and second parts weight r.*

As an illustration, we use the comb notation to verify a particular case of Conjecture 4.2 of [5], which arises in the study of the image of the Steenrod algebra action on $\mathbf{F}_2[x_1, \ldots, x_s]$. First, some terminology: Recall from Section 1 that given a positive integer f, $\mu(f)$ is defined to be the number of summands in the most efficient way of writing f as a sum of integers of the form $2^i - 1$. In [6], Singer observes that for any f there exists a unique set of integers $1 \le q_1 \le q_2 < q_3 < q_4 < \cdots < q_{\mu(f)} = [\log_2(f+1)]$ such that $f = \sum_{j=1}^{\mu(f)} (2^{q_j} - 1)$. Define the sequence $M_0(f) = (m_1, \ldots, m_{[\log_2(f+1)]})$ by letting m_i be the number of q_j's equal to i. Then $Sq(M_0(f))$ has excess $\mu(f)$, the minimal excess among Milnor basis elements of dimension f, which is to say among the summands of $\chi[Sq(f)]$.

For $s \ge 0$, let $M_s(f) = ((2^{s+1}-1)m_1, (2^{s+1}-1)m_2, \ldots, (2^{s+1}-1)m_{\mu(f)})$. Clearly $Sq(M_s(f))$ has dimension $(2^{s+1}-1)f$ and excess $(2^{s+1}-1)\mu(f)$. In Conjecture 3.3 it is claimed that $Sq(M_s(f))$ is a summand of an element of particular importance with respect to the $\mathcal{A}(2)$-action on $\mathbf{F}_2[x_1, \ldots, x_s]$ [5], and moreover that $(2^{s+1}-1)\mu(f)$ is minimal among the excesses of such summands:

Conjecture 3.3 (cf. Silverman [5]). For any integers $f \ge 1$ and $s \ge 0$,
(a) $Sq(M_s(f))$ is a summand of $\chi[Sq(2^s f) \cdot \ldots \cdot Sq(2f) \cdot Sq(f)]$.
(b) The excess $ex\{\chi[Sq(2^s f) \cdot \ldots \cdot Sq(2f) \cdot Sq(f)]\} = (2^{s+1}-1)\mu(f)$.

For the case $s = 0$, f arbitrary, Part (a) is immediate from Milnor's description of $\chi[Sq(f)]$ in [4], and Part (b) is exactly Proposition 6 of [2]. In view of the previous discussion, the case $s = 1$ may be restated as follows:

Conjecture. (a) The canonically bundled comb $C_b(M_1(f))$ of $M_1(f)$ admits an odd number of partitions whose first parts have weight $2f$.

(b) Let K be a comb of weight $3f$ in canonically bundled form, and suppose that K admits an odd number of partitions whose first parts have weight $2f$. Then $ex(K) \ge 3\mu(f)$. (Note that it follows immediately from the hypotheses that $ex(K) \ge \mu(3f)$ and $ex(K) \ge \mu(2f)$.)

The case $(s, f) = (1, 5)$ is shown below. To verify Part (a), observe that $C_b(M_1(f))$ admits exactly one partition whose first part has weight 10:

$$\begin{array}{cccc} 0 & 0 & 0 & \overline{1} \\ 0 & \overline{1} & 1 & 1 \\ \underline{1} & & \underline{1} & \end{array}.$$

To verify Part (b), observe that $\mu(2f) = 2$ and $3\mu(f) = 9$, so we need only check that the canonically bundled combs of weight 15 and excess 3, 5 and 7 all admit an even number of partitions whose first parts have weight 10. There are five such combs:

$$
\begin{array}{llllll}
T & (1,0,2) & (2,2,1) & (0,5) & (3,4) & (5,1,1) \\[1em]
K(T) &
\begin{array}{cc} 1 & 0 \\ & 1 \\ & 1 \\ & 1 \end{array} &
\begin{array}{ccc} 0 & 0 & 1 \\ 1 & 1 & 1 \\ & 1 & 1 \\ & & \end{array} &
\begin{array}{cc} 1 & 0 \\ 1 & 0 \\ & 1 \\ & 1 \end{array} &
\begin{array}{ccc} 1 & 0 & 0 \\ & 1 & 0 \\ & & 1 \\ & & 1 \end{array} &
\begin{array}{cccc} 1 & 0 & 1 & 1 \\ & 0 & 1 & 1 \\ & 1 & & 1 \\ & & & \end{array}
\end{array}
$$

Of these five, the first four admit no partitions of the desired form, and the fifth admits two,

$$
\begin{array}{cccc} \overline{1} & 0 & 1 & 1 \\ & \underline{0} & \underline{1} & 1 \\ & 1 & & \underline{1} \end{array}
\quad \text{and} \quad
\begin{array}{cccc} \overline{1} & 0 & 1 & 1 \\ & 0 & \underline{1} & \underline{1} \\ & \underline{1} & & 1 \end{array} .
$$

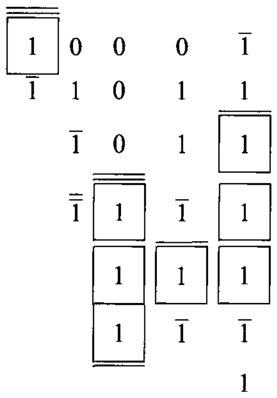
Thus Part (b) of the conjecture is verified for $(s, f) = (1, 5)$.

Using the analogous theory of combs whose teeth are ordered, one arrives at the following simpler if less tractable version of Part (b) of the conjecture for the case $s = 1$:

Conjecture. Let $f \geq 1$ be an integer and suppose that $H = (h_1, \ldots, h_t)$ is a t-tuple of positive integers such that $3f = \sum_{i=1}^{t}(2^{h_i} - 1)$. Let $\mathcal{L}(H)$ be the set of t-tuples $(l_1, \ldots, l_t)$ for which $0 \leq l_i \leq h_i$ for all i and $\sum_{i=1}^{t}(2^{l_i} - 1) = 2f$. Then the cardinality of $\mathcal{L}(H)$ is even if $t < 3\mu(f)$.

3.4. Products of several elements

The notion of partitions can easily be generalized to permit a characterization of the Milnor basis elements which appear as summands of a product $Sq(R_n) \cdot Sq(R_{n-1}) \cdot \ldots \cdot Sq(R_1)$. For $n \geq 2$, we define an *n-partitioned* comb K to be one in which each tooth is divided into n parts, some possibly empty, by $n-1$ horizontal lines. That is, to each generalized tooth τ of length l and size 2^σ is assigned an $(n-1)$-tuple of integers with $0 \leq \pi_1(\tau) \leq \pi_2(\tau) \leq \cdots \leq \pi_{n-1}(\tau) \leq l$. Let $\pi_0(\tau) = 0$ and $\pi_n(\tau) = l$ for all τ. The *ith part* of K, $1 \leq i \leq n$, is obtained by replacing all the 1's in each τ except those in rows $\sigma + \pi_{i-1}(\tau)$ through $\sigma + \pi_i(\tau) - 1$ with 0's. Thus 2-partitions are the familiar partitions of Section 2.3. In this picture, we show a 4-partitioned bundled comb whose third part is outlined:



Straightforward inductive arguments based on the reasoning of Sections 3.2 and 3.3 establish the following generalizations of Theorems 3.1 and 3.2 respectively:

Theorem 3.4. *With notation as above, the Milnor basis element $Sq(T)$ is a summand of $Sq(R_n)\cdot Sq(R_{n-1})\cdot\ldots\cdot Sq(R_1) \iff$ there is an odd number of n-partitions of $C_b(T)$ such that the ith part is a (bundled) comb for R_i for all i.*

Theorem 3.5. *Given any positive integers $w_1,\ldots,w_n$, the Milnor basis element $Sq(T)$ is a summand of $\chi[Sq(w_n)]\cdot\chi[Sq(w_{n-1})]\cdot\ldots\cdot\chi[Sq(w_1)] \iff$ there is an odd number of n-partitions of $C_b(T)$ such that the ith part is a (bundled) comb of weight w_i for all i.*

PART II. MANIPULATION OF ADEM RELATIONS

4. Coefficients in the Adem relations

Recall that in addition to the Milnor basis, $\mathcal{A}(2)$ has an additive basis consisting of admissible elements, i.e. products $Sq(a_1)\cdots Sq(a_n)$ with $a_r \geq 2a_{r+1}$ for $1 \leq r \leq n-1$ and $a_n > 0$ if $n > 1$. Inadmissible products are expressed in terms of admissible ones via the Adem relations [1]: if $a < 2b$ then

$$Sq(a)\cdot Sq(b) = \sum_{j=0}^{[a/2]} \binom{b-1-j}{a-2j} Sq(a+b-j)\cdot Sq(j). \tag{4}$$

We refer to the right-hand side of (4) as the *Adem expansion* of the inadmissible product $Sq(a)\cdot Sq(b)$. In Sections 4–6, we analyze the binomial coefficients in (4) and develop a method of computing whole families of related Adem expansions at once.

Recall from Section 1 that the binomial coefficient

$$\binom{n}{p} \text{ is odd} \iff n \succcurlyeq p.$$

For fixed a and b, let $J(a, b)$ be the set of integers j for which the coefficient

$$\binom{b-1-j}{a-2j}$$

in (4) is odd. In order to study the set $J(a, b)$, it will be convenient to think of this coefficient as determined not by the two variables a and b but rather by the single variable $b - [a/2] - 1$. We do so as follows. For positive integers F, define

$$\mathcal{K}(F) = \left\{ k \in 2\mathbf{N} \;:\; \binom{\frac{1}{2}k + F}{k} \equiv 1 \pmod 2 \right\};$$

$$\mathcal{K}^o(F) = \left\{ k \in \mathcal{K}(F) \;:\; \frac{1}{2}k + F \equiv 1 \pmod 2 \right\}.$$

The elements of these sets will be discussed in detail shortly.

Given an inadmissible Steenrod element $Sq(a) \cdot Sq(b)$, let

$$a' = \begin{cases} a & a \text{ even} \\ a-1 & a \text{ odd} \end{cases}$$

and define the integer $F = F(a, b) \geq 0$ by $F = b - a'/2 - 1$. Finally, let

$$\mathcal{K}_a(F) = \begin{cases} \{k \in \mathcal{K}(F) \;:\; k \leq a'\} & a \text{ even} \\ \{k \in \mathcal{K}^o(F) \;:\; k \leq a'\} & a \text{ odd.} \end{cases} \tag{5}$$

We claim that whatever the parity of a,

$$j \in J(a, b) \iff a' - 2j \in \mathcal{K}_a[F(a, b)], \tag{6}$$

so that (4) becomes

$$Sq(a) \cdot Sq(b) = \sum_{k \in \mathcal{K}_a[F(a,b)]} \Gamma_{a,b}(k) \tag{7}$$

where

$$\Gamma_{a,b}(k) \stackrel{\text{def}}{=} Sq\left(a + b - \frac{a'-k}{2}\right) \cdot Sq\left(\frac{a'-k}{2}\right).$$

Indeed, if a is even, we have

$$\binom{b-1-j}{a-2j} = \binom{\frac{a'-2j}{2} + F}{a'-2j},$$

so the claim follows immediately from the definitions of $J(a, b)$ and $\mathcal{K}_a(F)$. If a is odd, we have

$$\binom{b-1-j}{a-2j} = \binom{\frac{a'-2j}{2}+F}{a'+1-2j}.$$

As $a'+1-2j$ is odd, the requirement that the right-hand coefficient be odd amounts to requiring that both

$$\binom{\frac{a'-2j}{2}+F}{a'-2j} \quad \text{and} \quad \binom{\frac{a'-2j}{2}+F}{1}$$

be odd. These conditions along with $0 \le a'-2j \le a'$ say exactly that $a'-2j \in \mathcal{K}_a(F)$.

Eq. (7) implies that for fixed F, the Adem relations of inadmissible products $Sq(a)\cdot Sq(b)$ with $F(a, b) = F$ are closely related to each other. Indeed, the condition $\frac{1}{2}k + F \succcurlyeq k$ ensures that all elements of $\mathcal{K}(F)$ are $\le 2F$, so that $\mathcal{K}_a(F) = \mathcal{K}(F)$ for $a \ge 2F$. For a in that range, the Adem relations for all $Sq(a)\cdot Sq(b)$ with $F(a, b) = F$ have the same number of summands parametrized by the same values of k. One might think of such a as being in the "stable range" for F. As a decreases below $2F$, the summands $\Gamma_{a,b}(k)$ for larger values of $k \in \mathcal{K}_a(F)$, i.e. those summands with smaller second factors, cease to appear in the Adem expansions of $Sq(a)\cdot Sq(b)$.

5. The sets $\mathcal{K}(F)$ and $\mathcal{K}^o(F)$

We now develop notation for describing the sets of the previous section. For any integer G whose binary representation is $\gamma_\rho \cdots \gamma_1\gamma_0$, write $G^{[i]}$, $0 \le i \le \rho$ for the truncation $\gamma_i \cdots \gamma_1\gamma_0$. Let F and k be integers with binary representations $f_\tau f_{\tau-1} \cdots f_1 f_0$ and $\kappa_\sigma\kappa_{\sigma-1} \cdots \kappa_1\kappa_0$ respectively, and suppose that $k \in \mathcal{K}(F)$, so that $\frac{1}{2}k + F \succcurlyeq_i k$ for all $0 \le i \le \sigma$. Observe that

$$\frac{k}{2} + F \succcurlyeq_i k \Longleftrightarrow \kappa_{i+1} + f_i + \left[\frac{F^{[i-1]} + k^{[i]}/2}{2^i}\right] \succcurlyeq \kappa_i.$$

If $\kappa_i = 0$ this condition is empty; κ_{i+1} can be either 1 or 0. If $\kappa_i = 1$, then κ_{i+1} is determined by

$$\kappa_{i+1} \equiv f_i + \left[\frac{F^{[i-1]} + k^{[i]}/2}{2^i}\right] + 1 \pmod 2. \tag{8}$$

Let $1 \le p_1 < \cdots < p_{n_F} = \tau + 1$ be the set of indices for which $f_{p_l} \ne f_{p_l-1}$. One easily checks that $\sum_{i=g_l}^{p_l} 2^i \in \mathcal{K}(F)$ for $1 \le l \le n_F$ and $p_{l-1} + 1 \le g_l \le p_l$. In the statement and proof of Proposition 5.1, we write $S_{g_l}^{p_l}$ for $\sum_{i=g_l}^{p_l} 2^i$.

Proposition 5.1. *Each $k \in \mathcal{K}(F)$ is a linear combination $\sum_{l=1}^{n_F} \varepsilon_l S_{g_l}^{p_l}$ for some $\varepsilon_l = 0, 1$ and some $p_{l-1} + 1 \le g_l \le p_l$, with the condition that $g_l \ge p_{l-1} + 2$ when $\varepsilon_l = \varepsilon_{l-1} = 1$. (Of course g_l is unique if $\varepsilon_l \ne 0$.)*

Proof. Let $k \in \mathcal{K}(F)$. We prove by induction on $l \le n_F$ that for $1 \le i \le l$ there exist g_i and ε_i such that $k^{[p_l]} = \sum_{i=1}^{l} \varepsilon_i S_{g_i}^{p_i}$ (resp. $k^{[p_l+1]} = \sum_{i=1}^{l} \varepsilon_i S_{g_i}^{p_i}$ if $\varepsilon_l = 1$), and moreover that

$$\left[\frac{F^{[p_l]} + k^{[p_l+1]}/2}{2^{p_l+1}}\right] = 0.$$

Case $l = 1$: Let $k \in \mathcal{K}(F)$. If $k \equiv 0 \pmod{2^{p_1+1}}$, then take $\varepsilon_1 = 0$ and $g_1 = p_1$. Otherwise there exists g_1, $1 \le g_1 \le p_1$, such that $\kappa_j = 0$, $0 \le j < g_1$ and $\kappa_{g_1} = 1$. We claim that $k^{[p_1+1]} = S_{g_1}^{p_1}$, i.e. that $\kappa_j = 1$ for $g_1 \le j \le p_1$ and that $\kappa_{p_1+1} = 0$. We show first that whether $f_{p_1} = 1$ or 0, we have

$$\left[\frac{F^{[j-1]} + k^{[j]}/2}{2^j}\right] \equiv_2 f_{p_1} + 1 \tag{9}$$

for $g_1 \le j \le p_1$, where $\equiv_2$ denotes congruence modulo 2. Indeed, whatever the remaining κ_m, $g_1 + 1 \le m \le p_1$ may turn out to be, we always have $2^{g_1} \le k^{[j]} \le 2^{j+1} - 2^{g_1}$. If $f_{p_1} = 0$, then $f_m = 1$ for $m < p_1$, so $F^{[j-1]} = 2^j - 1$ for $g_1 \le j \le p_1$. Thus $F^{[j-1]} + k^{[j]}/2 \ge 2^j - 1 + 2^{g_1-1} \ge 2^j$. On the other hand, if $f_{p_1} = 1$ then $F^{[j-1]} = 0$, so $F^{[j-1]} + k^{[j]}/2 \le (2^{j+1} - 2^{g_1})/2 \le 2^j$.

We now prove the claim. If $g_1 = p_1$, then (8) and (9) give

$$\kappa_{p_1+1} \equiv_2 f_{p_1} + \left[\frac{F^{[p_1-1]} + k^{[p_1]}/2}{2^{p_1}}\right] + 1 \equiv_2 0$$

as desired. If $g_1 < p_1$, then $f_j \equiv_2 1 + f_{p_1}$ for $g_1 \le j \le p_1 - 1$. Repeated application of (8) and (9) gives

$$\begin{aligned} \kappa_{j+1} &\equiv_2 f_j + \left[\frac{F^{[j-1]} + k^{[j]}/2}{2^j}\right] + 1 \\ &\equiv_2 1 + f_{p_l} + \left[\frac{F^{[j-1]} + k^{[j]}/2}{2^j}\right] + 1 \equiv_2 1 \quad g_1 \le j \le p_1 - 1; \\ \kappa_{p_1+1} &\equiv_2 0. \end{aligned} \tag{10}$$

Whether $f_{p_1} = 1$ or 0, we have $F^{[p_1]} + k^{[p_1+1]}/2 \le 2^{p_1} + (2^{p_1} - 2^{g_1-1}) < 2^{p_1+1}$, so

$$\left[\frac{F^{[p_1]} + k^{[p_1+1]}/2}{2^{p_1+1}}\right] = 0.$$

This proves the inductive claim for $l = 1$.

General Case: Now let $k \in \mathcal{K}(F)$ and $2 \le l \le n$, and assume inductively that there exist ε_i and g_i such that $k^{[p_l]} = \sum_{i=1}^{l} \varepsilon_i S_{g_i}^{p_i}$ (resp. $k^{[p_l+1]} = \sum_{i=1}^{l} \varepsilon_i S_{g_i}^{p_i}$ if $\varepsilon_l = 1$). Assume moreover that

$$\left[\frac{F^{[p_l]} + k^{[p_l+1]}/2}{2^{p_l+1}}\right] = 0.$$

If $\kappa_j = 0$ for $p_l + 1 \leq j \leq p_{l+1}$, we take $\varepsilon_{l+1} = 0$ and $g_{l+1} = p_{l+1}$. Otherwise there exists g_{l+1}, $p_l + 1 \leq g_{l+1} \leq p_{l+1}$ (resp. $p_l + 2 \leq g_{l+1} \leq p_{l+1}$) such that $\kappa_j = 0$, $p_l + 1 \leq j < g_{l+1}$ (resp. $p_l + 2 \leq j < g_{l+1}$) and $\kappa_{g_{l+1}} = 1$.

We show as before that

$$\left[\frac{F^{[j-1]} + k^{[j]}/2}{2^j}\right] \equiv_2 1 + f_{p_{l+1}} \quad \text{for } g_{l+1} \leq j \leq p_{l+1};$$

the rest of the argument proceeds as in the case $l = 1$. Write

$$F^{[j-1]} + k^{[j]}/2 = \sum_{m=p_l+1}^{j-1} f_m 2^m + \sum_{m=p_l+1}^{j-1} \kappa_{m+1} 2^m + F^{[p_l]} + k^{[p_l+1]}/2;$$

as both the summations are divisible by 2^{p_l+1} and $F^{[p_l]} + k^{[p_l+1]}/2 < 2^{p_l+1}$ by the inductive hypothesis, we have

$$\left[\frac{F^{[j-1]} + k^{[j]}/2}{2^j}\right] = \left[\frac{\sum_{m=p_l+1}^{j-1} f_m 2^m + \sum_{m=p_l+1}^{j-1} \kappa_{m+1} 2^m}{2^j}\right].$$

Now $2^{g_{l+1}-1} \leq \sum_{m=p_l+1}^{j-1} \kappa_{m+1} 2^m \leq S_{g_{l+1}-1}^{p_{l+1}-1}$ for $g_{l+1} \leq j \leq p_{l+1}$ regardless of what the κ_m turn out to be. If $f_{p_{l+1}} = 0$, then $\sum_{m=p_l+1}^{j-1} f_m 2^m = S_{p_l+1}^{j-1}$, so

$$\left[\frac{\sum_{m=p_l+1}^{j-1} f_m 2^m + \sum_{m=p_l+1}^{j-1} \kappa_{m+1} 2^m}{2^j}\right] \geq \left[\frac{S_{p_l+1}^{j-1} + 2^{g_{l+1}}}{2^j}\right] = 1.$$

If on the other hand $f_{p_{l+1}} = 1$, then $\sum_{m=p_l+1}^{j-1} f_m 2^m = 0$, so

$$\left[\frac{\sum_{m=p_l+1}^{j-1} f_m 2^m + \sum_{m=p_l+1}^{j-1} \kappa_{m+1} 2^m}{2^j}\right] = \left[\frac{\sum_{m=p_l+1}^{j-1} \kappa_{m+1} 2^m}{2^j}\right] \leq \left[\frac{S_{g_{l+1}-1}^{j-1}}{2^j}\right] = 0.$$

The argument from the case $l = 1$ now applies to show that $\kappa_j = 1$, $g_{l+1} \leq j \leq p_{l+1}$ and $\kappa_{p_{l+1}+1} = 0$.

Finally, whether $f_{p_{l+1}} = 1$ or 0 we have

$$\begin{aligned}\left[\frac{F^{[p_{l+1}]} + k^{[p_{l+1}+1]}/2}{2^{p_{l+1}+1}}\right] &= \left[\frac{\sum_{m=p_l+1}^{p_{l+1}} f_m 2^m + \sum_{m=p_l+1}^{p_{l+1}} \kappa_{m+1} 2^m}{2^{p_{l+1}+1}}\right] \\ &\leq \left[\frac{2^{p_{l+1}} + S_{g_{l+1}-1}^{p_{l+1}-1}}{2^{p_{l+1}+1}}\right] = 0.\end{aligned}$$

This completes the inductive step and proves the proposition. □

As $\mathcal{K}^o(F) = \{k \in \mathcal{K}(F) \ : \ \frac{1}{2}k + F \equiv 1 \pmod 2\}$, we have the following corollary:

Corollary 5.2. *With notation as above,*

$$\mathcal{K}^o(F) = \begin{cases} \{\sum_{l=1}^{n} \varepsilon_l S_{g_l}^{p_l} \ : \ g_1 \geq 2 \text{ if } \varepsilon_1 = 1\} & F \equiv 1 \pmod 2 \\ \{\sum_{l=1}^{n} \varepsilon_l S_{g_l}^{p_l} \ : \ \varepsilon_1 = 1, \gamma_1 = 1\} & F \equiv 0 \pmod 2. \end{cases}$$

6. Computation of Adem relations

Proposition 5.1 suggests that the fewer the alternations between 1's and 0's in the binary representation of F, the simpler the Adem expansions of inadmissible products $Sq(a)\cdot Sq(b)$ with $F(a,b) = F$. As an illustration of the technique we verify two families of Adem relations in which the F's are of the form $2^i - 1 = \overbrace{111\cdots 11}^{i}$. The first of these generalizes the familiar relation $Sq(2^\alpha)\cdot Sq(2^\alpha) = \sum_{\beta=0}^{\alpha-1} Sq(\sum_{i=\beta}^{\alpha} 2^i)\cdot Sq(2^\beta)$; the second will be used in Section 7 to put constraints on the Milnor basis elements occurring as summands in $\chi[Sq(2f)\cdot Sq(f)]$. In what follows, we use the conventions that $\sum_{i=n}^{m} 2^i = 0$ if $n > m$ and $Sq(n) = 0$ $(\neq Sq(0))$ if $n < 0$.

Theorem 6.1. *Let $\alpha \geq 0$ and c be integers. Then*

$$Sq(2c + 2^\alpha)\cdot Sq(c + 2^\alpha) = \sum_{\beta=0}^{\alpha-1} Sq(2c + \textstyle\sum_{i=\beta}^{\alpha} 2^i)\cdot Sq(c + 2^\beta). \tag{11}$$

Proof. Let $a = 2c + 2^\alpha$ and $b = c + 2^\alpha$.

Case $\alpha = 0$: In this case $a = 2b - 1$, and it is well known (or easily verified using the above techniques) that $Sq(2b-1)\cdot Sq(b) = 0$ for all b.

Case $\alpha = 1$: In this case $F = F(a,b) = c + 1 - (c+1) = 0$, so the only possible values for k and $2j$ are 0 and $2c + 2^2$ respectively. Thus $Sq(2c + 2^2)\cdot Sq(c+2) = Sq(2c+3)\cdot Sq(c+1)$ as claimed.

Case $\alpha \geq 2$: In this case a is even and $F = 2^{\alpha-1} - 1$, so $a \geq 2F$ and $\mathcal{K}_a(F) = \mathcal{K}(F)$. From Proposition 5.1 and Eq. (6), we have that the possible values for $k \in \mathcal{K}_a(F)$ and $j \in J$ are

$$k = \sum_{i=\beta}^{\alpha-1} 2^i, \quad 1 \leq \beta \leq \alpha \quad \text{and} \quad 2j = 2c + 2^\beta, \quad 1 \leq \beta \leq \alpha. \tag{12}$$

These values of $2j$ correspond exactly to the summands on the right in (11). □

The left-hand factor $Sq(2c + 2^\alpha)$ of the inadmissible product considered in Theorem 6.1 is in the stable range for $F = 2^{\alpha-1} - 1$, and so the summands of the Adem expansion may be described in terms of a parameter β which depends on α alone. In the next computation the left-hand factor is not in the appropriate stable range, so the summands of the expansion depend on that factor as well as on α.

Given any integer f, define integers $\Lambda(f)$ and $\lambda(f)$ as follows:

$$\Lambda(f) = [\log_2(f+1)]$$

$$\lambda(f) = \begin{cases} \Lambda(f) + 1, & f = 2^{\Lambda(f)} - 1 \\ \min\{M \ : \ f \geq \sum_{i=M}^{\Lambda(f)} 2^i\} & f \text{ not of the form } 2^n - 1. \end{cases}$$

That is, $2^{\Lambda(f)} - 1 \leq f \leq 2^{\Lambda(f)+1} - 2$, and if $f \neq 2^{\Lambda(f)} - 1$ then the coefficients of the binary representation $f = \sum_{i=0}^{\Lambda(f)} \varepsilon_i 2^i$ satisfy

$$\varepsilon_i = \begin{cases} 1 & \lambda(f) \le i \le \Lambda(f), \\ 0 & i = \lambda(f) - 1. \end{cases}.$$

When f is fixed, we write Λ and λ for $\Lambda(f)$ and $\lambda(f)$.

Theorem 6.2. *The following Adem relation holds for any integer* $f \ge 1$:

$$Sq(2f - \textstyle\sum_{i=0}^{\Lambda} 2^i) \cdot Sq(f) = \displaystyle\sum_{\pi=\lambda}^{\Lambda} Sq(2f - \textstyle\sum_{i=0}^{\pi-1} 2^i) \cdot Sq(f - \sum_{i=\pi}^{\Lambda} 2^i). \tag{13}$$

Proof

Case $f = 2^\Lambda - 1$: In this case both sides of (13) are 0 by convention.
Case $f \ne 2^\Lambda - 1$: This time $a = 2f - \sum_{i=0}^{\Lambda} 2^i$ is odd, and

$$F = F(a, f) = f - 1 - \frac{a-1}{2} = 2^\Lambda - 1.$$

The possible values of $k \in \mathcal{K}(F)$ are

$$k = \sum_{i=\beta}^{\Lambda} 2^i, \quad 1 \le \beta \le \Lambda + 1.$$

By the definition of Λ we have

$$a = 2f - (2^{\Lambda+1} - 1) \le 2(2^{\Lambda+1} - 2) - (2^{\Lambda+1} - 1) < 2F$$

so a is not in the stable range for F; one easily checks that

$$\mathcal{K}_a(F) = \{\textstyle\sum_{i=\beta}^{\Lambda} 2^i \ : \ \lambda + 1 \le \beta \le \Lambda + 1\}.$$

Thus the possible values of $2j$ are $2f - \sum_{i=\beta}^{\Lambda+1}$, $\lambda + 1 \le \beta \le \Lambda+1$. These values correspond exactly to the summands on the right in (13). $\square$

Note. One may regard the terms on either side of (13) as representing all possible ways to split the sum $\sum_{i=0}^{\Lambda} 2^i$ as $\sum_{i=0}^{\pi-1} 2^i + \sum_{i=\pi}^{\Lambda} 2^i$ in such a way that the differences $2f - \sum_{i=0}^{\pi-1} 2^i$ and $f - \sum_{i=\pi}^{\Lambda} 2^i$ are both positive. Only when the entire sum is subtracted from $2f$ is the associated product inadmissible.

Using the techniques of this section and exploiting the relationship between the Adem expansions of inadmissible products $Sq(a) \cdot Sq(b)$ and $Sq(2a) \cdot Sq(2b)$, one may with difficulty prove the following generalization of Theorem 6.2. In the statements of Theorem 6.3 and Conjecture 6.4, we continue to use the conventions that $\sum_{i=n}^{m} 2^i = 0$ if $n > m$ and $Sq(n) = 0$ if $n < 0$. Both the theorem and the conjecture may be interpreted along the lines of the note following the proof of Theorem 6.2.

Theorem 6.3. *Let $f \geq 1$ and define $\Lambda = \Lambda(f)$ and $\lambda = \lambda(f)$ as in Theorem 6.2. Then for all $s \geq 1$, the following equality holds:*

$$\left(\sum_{\rho=\lambda+s-1}^{\Lambda+s} Sq(2^s f - \textstyle\sum_{k=s-1}^{\rho-1} 2^k) \cdot Sq(2^{s-1} f - \sum_{k=\rho}^{\Lambda+s-1} 2^k)\right)$$
$$\cdot Sq(2^{s-2} f) \cdot Sq(2^{s-3} f) \cdot \ldots \cdot Sq(2f) \cdot Sq(f) = 0.$$

It seems likely that the same methods could be used to establish yet a further generalization of Theorem 6.2, but so far the bookkeeping difficulties have proven insurmountable. We state this generalization as a conjecture:

Conjecture 6.4. With notation as above, let $0 \leq w \leq v \leq s$. Then

$$\Big(\sum Sq(2^s f - \textstyle\sum_{i=w}^{\rho_1-1} 2^i) \cdot Sq(2^{s-1} f - \sum_{i=\rho_1}^{\rho_2-1} 2^i) \cdot Sq(2^{s-2} f - \sum_{i=\rho_2}^{\rho_3-1} 2^i)$$
$$\cdot \ldots \cdot Sq(2^v f - \textstyle\sum_{i=\rho_{s-v}}^{\Lambda+v} 2^i)\Big) Sq(2^{v-1} f) \cdot Sq(2^{v-2} f) \cdot \ldots \cdot Sq(f) = 0,$$

where the left-most sum is over all sequences $w \leq \rho_1 \leq \rho_2 \leq \cdots \leq \rho_{s-v} \leq \Lambda + v + 1$.

PART III. APPLICATION

7. Summands of $\chi[Sq(2^s f) \cdot \ldots \cdot Sq(2f) \cdot Sq(f)]$

In this section, we use the techniques developed in Parts I and II to prove a special case of the following conjecture (cf. Conjecture 3.3). Recall from Section 6 the notation $\Lambda = \Lambda(f) = [\log_2(f + 1)]$.

Conjecture 7.1. Let $s \geq 0$ and $f \geq 1$ be integers, and suppose that $Sq(r_1, \ldots, r_n)$ is a summand of $\chi[Sq(2^s f) \cdot \ldots \cdot Sq(2f) \cdot Sq(f)]$. Then $r_i = 0$ for $i \geq \Lambda + 1$.

In view of Theorem 3.5, the conjecture may be restated in terms of combs as follows:

Conjecture 7.1′. Let $s \geq 0$ and $f \geq 1$ be integers. Let $R = (r_1, \ldots, r_n)$ with $\sum_{i=1}^n r_i(2^i - 1) = (2^{s+1} - 1)f$, and suppose that the canonically bundled comb $C_b(R)$ admits an odd number of $(s + 1)$-partitions whose ith parts have weight $2^{i-1} f$ for $1 \leq i \leq s + 1$. Then $r_i = 0$ for $i \geq \Lambda + 1$.

That is to say, the combs associated to summands of $\chi[Sq(2^s f) \cdot \ldots \cdot Sq(2f) \cdot Sq(f)]$ are conjectured to be no longer in the tooth than are those associated to summands of $\chi[Sq(f)]$.

Theorem 7.2. *For all $f \geq 1$, Conjectures 7.1 and 7.1′ are true when $s = 0$ and $s = 1$.*

Proof. The conjecture for $s = 0$, f arbitrary is true for dimension reasons. In the proof for $(1, f)$ we identify $C(T)$ with $Sq(T)$ for all $T = (t_1, \ldots, t_n)$, so that in particular addition is defined on combs, and we may speak of combs as summands of $\chi[Sq(2f)\cdot Sq(f)]$.

Suppose then that $C(R)$ is a summand of $\chi[Sq(2f)\cdot Sq(f)]$. Its weight satisfies $\sum(2^i-1)r_i = 3b < 3(2^{\Lambda+1}-1)$, so that $r_i = 0$ for $i \geq \Lambda+3$ and $r_{\Lambda+2} \leq 1$, $r_{\Lambda+1} \leq 2$. We show first that $r_{\Lambda+2} = 0$. Suppose on the contrary that $r_{\Lambda+2} = 1$, so that the canonical comb $C_b(R)$ has a tooth $\tau^{\Lambda+2}$ of length $\Lambda+2$ and size 1. Choose any 2-partition of $C_b(R)$ such that the first part has weight $2f$ and the second part f. If the partition number $\pi(\tau^{\Lambda+2})$ of $\tau^{\Lambda+2}$ is $\leq \Lambda+1$, then the second part of the partition has weight $\geq 2^{\Lambda+1} > f$, which is impossible. On the other hand, if $\pi(\tau^{\Lambda+2}) = \Lambda+2$, then the first part of the partition has weight $\geq 2^{\Lambda+2}-1 > 2f$, which is also impossible. Thus there are no 2-partitions of $C_b(R)$ for which the first part has weight $2f$, and so by Theorem 3.2 $Sq(R)$ is not a summand of $\chi[Sq(2f)\cdot Sq(f)]$. This contradicts the assumption, so $r_{\Lambda+2}$ must be 0 after all.

The same argument may be used to show that

$$r_{\Lambda+1}\begin{cases} =0 & f=2^{\Lambda}-1, \\ \leq 1 & f\neq 2^{\Lambda}-1. \end{cases}$$

This proves the case $s=1$ for f of the form $2^{\Lambda}-1$ and leaves only the possibility that f is not of this form and $r_{\Lambda+1}=1$. In order to eliminate this possibility, we introduce the following notation.

Let $\tau^{l,\sigma}$ (resp. $\tau_\pi^{l,\sigma}$) denote a tooth of length l and size 2^σ (resp. with partition number π). As before, we write τ_π^l for $\tau_\pi^{l,0}$. Given any comb C, let $C\oplus\tau^{l,\sigma}$ be the comb obtained from C by adding a tooth of length l and size 2^σ; if C is an ordinary (i.e. unbundled) comb and $\sigma=0$, then $C\oplus\tau^l$ is also an ordinary comb. If C already has a tooth of length l and size 2^σ, then write $C\ominus\tau^{l,\sigma}$ for the comb obtained by extracting that tooth. $C^{\mathrm{p}}\oplus\tau_\pi^{l,\sigma}$ and $C^{\mathrm{p}}\ominus\tau_\pi^{l,\sigma}$ are defined analogously for partitioned combs C^{p}.

Given any two combs C_1 and C_2, evidently $C_1\oplus\tau^{l,\sigma}=C_2\oplus\tau^{l,\sigma} \iff C_1=C_2$. More generally, for any set $\{C_1,\dots,C_n\}$ of (non-trivial) combs, we have $\sum_{i=1}^n(C_i\oplus\tau^{l,\sigma})=0 \iff \sum_{i=1}^n C_i=0$.

The underlying comb (resp. underlying bundled comb) of a partitioned bundled comb C^{p} is written UC^{p} (resp. U_bC^{p}). In particular, $U(C^{\mathrm{p}}\oplus\tau_\pi^l)=UC^{\mathrm{p}}\oplus\tau^l$.

Let $\mathcal{R}^i$, $i=0,1$, be the set of canonically bundled combs of weight $3f$ having $r_{\Lambda+1}=2^i$, so that for dimension reasons each $C\in\mathcal{R}^i$ has a tooth $\tau^{\Lambda+1,i}$ of length $\Lambda+1$ and size 2^i and no other generalized teeth of length $\geq \Lambda+1$. Let $\mathcal{R}^{P,i}$ be the set of 2-partitioned combs C^{p} with $UC^{\mathrm{p}}\in\mathcal{R}^i$ whose second parts have weight f. By Theorem 3.2, the summands $C(R')$ of $\chi[Sq(2f)\cdot Sq(f)]$ having $r'_{\Lambda+1}=1$ add up to $\sum_{C^{\mathrm{p}}\in\mathcal{R}^{P,0}}UC^{\mathrm{p}}$. To prove Theorem 7.2, it therefore suffices to prove that

$$\sum_{C^{\mathrm{p}}\in\mathcal{R}^{P,0}} UC^{\mathrm{p}}=0. \tag{14}$$

In order to study this sum, it is useful to sort the elements of $\mathcal{R}^{P,i}$ according to the partition number of the long tooth: for $0 \le \pi \le \Lambda+1$, let $\mathcal{R}^{P,i}_{\pi} = \{C^{\mathrm{p}} \in \mathcal{R}^{P,i} : \pi(\tau^{\Lambda+1,i}) = \pi\}$. Observe that $\mathcal{R}^{P,0}_{\pi} = \emptyset$ for $0 \le \pi \le \lambda-1$, where $\lambda = \lambda(f) = \min\{M : f \ge \sum_{i=M}^{\Lambda(f)} 2^i\}$ as in Section 6. Finally, for $\lambda \le \pi \le \Lambda+1$ let $\mathcal{S}^P(\pi)$ be the set of 2-partitioned canonically bundled combs whose first parts have weight $2f - \sum_{k=0}^{\pi-1} 2^k$ and second parts $f - \sum_{k=\pi}^{\Lambda} 2^k$.

To each $C^{\mathrm{p}} \in \mathcal{R}^{P,0}_{\pi}$ is associated the 2-partitioned bundled comb $C^{\mathrm{p}} \ominus \tau^{\Lambda+1}_{\pi} \in \mathcal{S}^P(\pi)$, which, as observed above, has no teeth of length $\ge \Lambda+1$. Conversely, each $K^{\mathrm{p}} \in \mathcal{S}^P(\pi)$ having no teeth of length $\ge \Lambda+1$ determines the 2-partitioned canonically bundled comb $K^{\mathrm{p}} \oplus \tau^{\Lambda+1}_{\pi} \in \mathcal{R}^{P,0}_{\pi}$. The remaining elements of $\mathcal{S}^P(\pi)$ – if there are any; there are none when $3b < 2(2^{\Lambda+1}-1)$ – have for dimension reasons exactly one tooth of length $\Lambda+1$, so are classified by the partition number of this tooth: for $0 \le \rho \le \Lambda+1$, define $\mathcal{S}^P_{\rho}(\pi) = \{K^{\mathrm{p}} \in \mathcal{S}^P(\pi) : \pi(\tau^{\Lambda+1}) = \rho\}$. We now have

$$
\begin{aligned}
\sum_{C^{\mathrm{p}} \in \mathcal{R}^{P,0}} UC^{\mathrm{p}} &= \sum_{\pi=\lambda}^{\Lambda+1} \sum_{C^{\mathrm{p}} \in \mathcal{R}^{P,0}_{\pi}} UC^{\mathrm{p}} \\
&= \sum_{\pi=\lambda}^{\Lambda+1} \sum_{K^{\mathrm{p}} \in \mathcal{S}^P(\pi) \setminus \left[\coprod_{\rho=0}^{\Lambda+1} \mathcal{S}^P_{\rho}(\pi)\right]} U(K^{\mathrm{p}} \oplus \tau^{\Lambda+1}_{\pi}) \\
&= \sum_{\pi=\lambda}^{\Lambda+1} \sum_{K^{\mathrm{p}} \in \mathcal{S}^P(\pi)} U(K^{\mathrm{p}} \oplus \tau^{\Lambda+1}_{\pi}) \\
&\quad + \sum_{\pi=\lambda}^{\Lambda+1} \sum_{K^{\mathrm{p}} \in \mathcal{S}^P_{\pi}(\pi)} U(K^{\mathrm{p}} \oplus \tau^{\Lambda+1}_{\pi}) + \sum_{\pi=\lambda}^{\Lambda+1} \sum_{\rho \ne \pi} \sum_{K^{\mathrm{p}} \in \mathcal{S}^P_{\rho}(\pi)} U(K^{\mathrm{p}} \oplus \tau^{\Lambda+1}_{\pi}). \quad (15)
\end{aligned}
$$

We proceed to show that each of these double sums vanishes.

Since $U(K^{\mathrm{p}} \oplus \tau^{\Lambda+1}_{\pi}) = U(K^{\mathrm{p}}) \oplus \tau^{\Lambda+1}$, the first double sum in (15) vanishes if $\sum_{\pi=\lambda}^{\Lambda+1} \sum_{K^{\mathrm{p}} \in \mathcal{S}^P(\pi)} U(K^{\mathrm{p}})$ does. By Theorem 3.2, we have

$$
\sum_{K^{\mathrm{p}} \in \mathcal{S}^P(\pi)} U(K^{\mathrm{p}}) = \chi \left[Sq\left(2f - \sum_{k=0}^{\pi-1} 2^k\right) \cdot Sq\left(f - \sum_{k=\pi}^{\Lambda} 2^k\right) \right]
$$

for $\lambda \le \pi \le \Lambda+1$, after the identification of $C(T)$ with $Sq(T)$. Thus

$$
\begin{aligned}
\sum_{\pi=\lambda}^{\Lambda+1} \sum_{K^{\mathrm{p}} \in \mathcal{S}^P(\pi)} U(K^{\mathrm{p}}) &= \sum_{\pi=\lambda}^{\Lambda+1} \chi \left[Sq\left(2f - \sum_{k=0}^{\pi-1} 2^k\right) \cdot Sq\left(f - \sum_{k=\pi}^{\Lambda} 2^k\right) \right] \\
&= \chi \left[\sum_{\pi=\lambda}^{\Lambda+1} Sq\left(2f - \sum_{k=0}^{\pi-1} 2^k\right) \cdot Sq\left(f - \sum_{k=\pi}^{\Lambda} 2^k\right) \right]. \quad (16)
\end{aligned}
$$

But Theorem 6.2 of Part II implies that the sum in (16) vanishes. Thus the first double sum in (15) vanishes as well.

As for the vanishing of the second double sum, observe that if $K^{\mathrm{p}} \in \mathcal{S}_\pi^P(\pi)$, then $K^{\mathrm{p}} \ominus \tau_\pi^{\Lambda+1}$ is a 2-partitioned canonically bundled comb all of whose teeth have length $\leq \Lambda$. Thus the comb $K_1 \stackrel{\text{def}}{=} U_{\mathrm{b}}(K^{\mathrm{p}} \ominus \tau_\pi^{\Lambda+1}) \oplus \tau^{\Lambda+1,1}$ is also in canonically bundled form and in fact is an element of $\mathcal{R}^{P,1}$. Moreover, the partition structure of $K^{\mathrm{p}} \oplus \tau_\pi^{\Lambda+1} = (K^{\mathrm{p}} \ominus \tau_\pi^{\Lambda+1} \oplus \tau_\pi^{\Lambda+1}) \oplus \tau_\pi^{\Lambda+1}$ induces the partition structure $K_1^{\mathrm{p}} = (K^{\mathrm{p}} \ominus \tau_\pi^{\Lambda+1}) \oplus \tau_\pi^{\Lambda+1,1}$ on K_1, and $K_1^{\mathrm{p}} \in \mathcal{R}_\pi^{P,1}$. One checks easily that the map

$$\coprod_\pi \{K^{\mathrm{p}} \oplus \tau_\pi^{\Lambda+1} \ : \ K^{\mathrm{p}} \in \mathcal{S}_\pi^P(\pi)\} \longrightarrow \coprod_\pi \mathcal{R}_\pi^{P,1}$$

$$K^{\mathrm{p}} \oplus \tau_\pi^{\Lambda+1} \longmapsto (K^{\mathrm{p}} \ominus \tau_\pi^{\Lambda+1}) \oplus \tau_\pi^{\Lambda+1,1}$$

is a bijection in which corresponding elements have the same underlying (ordinary) combs. Thus the second double sum in (15) satisfies

$$\sum_{\pi=\lambda}^{\Lambda+1} \sum_{K^{\mathrm{p}} \in \mathcal{S}_\pi^{\mathrm{p}}(\pi)} U(K^{\mathrm{p}} \oplus \tau_\pi^{\Lambda+1}) = \sum_{\pi=\lambda}^{\Lambda+1} \sum_{C^{\mathrm{p}} \in \mathcal{R}_\pi^{P,1}} U(C^{\mathrm{p}}).$$

But the double sum on the right represents the sum of the Milnor basis elements $Sq(t_1, \dots, t_{\Lambda+1})$ appearing in $\chi[Sq(2f) \cdot Sq(f)]$ for which $t_{\Lambda+1} = 2$, and as we saw above there are no such elements. Thus the second double sum in (15) vanishes, and it remains only to show that the third double sum vanishes as well.

Suppose $K^{\mathrm{p}} \in \mathcal{S}_\rho^P(\pi)$ for some $\rho \neq \pi$. As before, $K^{\mathrm{p}} \ominus \tau_\rho^{\Lambda+1}$ is a 2-partitioned canonically bundled comb with teeth of length $\leq \Lambda$. This time we have, for each (π, ρ), a bijection

$$\mathcal{S}_\rho^P(\pi) \stackrel{\iota}{\longrightarrow} \mathcal{S}_\pi^P(\rho)$$

$$K^{\mathrm{p}} \longmapsto (K^{\mathrm{p}} \ominus \tau_\rho^{\Lambda+1}) \oplus \tau_\pi^{\Lambda+1}$$

which amounts to changing the partition number of the tooth of length $\Lambda+1$ from ρ to π. Again, $U(K^{\mathrm{p}}) = U(\iota K^{\mathrm{p}})$, and so $U(K^{\mathrm{p}} \oplus \tau_\pi^{\Lambda+1}) = U(K^{\mathrm{p}}) \oplus \tau^{\Lambda+1} = U(\iota K^{\mathrm{p}} \oplus \tau_\rho^{\Lambda+1})$. The third double sum in (15) therefore satisfies

$$\sum_{\pi=\lambda}^{\Lambda+1} \sum_{\rho \neq \pi} \sum_{K^{\mathrm{p}} \in \mathcal{S}^P(j)} U(K^{\mathrm{p}} \oplus \tau_\pi^{\Lambda+1})$$

$$= \sum_{\pi=\lambda}^{\Lambda+1} \sum_{\rho < \pi} \sum_{K^{\mathrm{p}} \in \mathcal{S}^P(j)} [U(K^{\mathrm{p}} \oplus \tau_\pi^{\Lambda+1}) + U(\iota K^{\mathrm{p}} \oplus \tau_\rho^{\Lambda+1})]$$

$$= 0.$$

Thus the third double sum in (15) vanishes along with the other two, and so $\sum_{C^{\mathrm{p}} \in \mathcal{R}^{P,0}} UC^{\mathrm{p}} = 0$. By (14), this completes the proof of Theorem 7.2.

The proof that the first double sum in (15) vanished depended on Theorem 6.2. The generalization Theorem 6.3 leads to a stronger result:

Theorem 7.3. *Let* $s \geq 2$ *and* $f \geq 1$ *be integers, and let* $\Lambda = [\log_2(f+1)]$. *If* $Sq(R)$ *is a summand of* $\chi[Sq(2^s f) \cdot \ldots \cdot Sq(f)]$, *then*

$$r_{\Lambda+t} \begin{cases} \leq 2^{s-t} - 1, & 1 \leq t \leq s-1 \\ = 0, & t \geq s. \end{cases}$$

Conjecture 7.1 itself would follow by a similar argument from Conjecture 6.4.

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